A posteriori error estimators for A and Ω magnetostatic formulations based on equilibrated fluxes reconstructions

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In this paper, the equilibrated type a posteriori error estimates based on flux reconstructions by local problems are introduced to evaluate the discretization error in the finite element computation for both cases of A and Ω formulations. A comparison with residual type error estimator, as well as the classical equilibrated error estimator based on the non-verification of constitutive laws are carried out in an academic example. Our proposed estimators are shown to be more efficiency to the correspondent discretization errors.

Index Terms—A posteriori estimator, finite element method (FEM), magnetostatic problem, potential formulation.

I. INTRODUCTION

IN order to improve the quality of the numerical results of finite element computation in electromagnetic problems, it is necessary to quantify the discretization error which is linked to the mesh. A posteriori error estimates are widely developed in magnetostatic problems to evaluate the discretization error [1], [2]. Some comparisons have been done between different error estimators [3], [4]. In the previous work, it is shown that, the residual type estimator [2] which evaluates respectively the equations in each element, the discontinuities of fields on the interfaces of element, and the verification of the boundary condition, can give a relative error distribution, but cannot give a guaranteed bound for the global error. On the other hand, the classical equilibrated estimator [1] which is based on the non-verification of the constitutive laws, can give a guaranteed bound for the sum of errors from two potential formulations, but cannot give a correspondent error distributions to each formulation. In this work, we propose an improved equilibrated error estimator based on the flux reconstruction technical on local problem [5], [6] for both potential formulations, which can be provided reliable and efficiency. Moreover, it take the advantages of both residual one and classical equilibrated one, i.e. for each formulation, proposed estimator can give a global guaranteed error bound, as well as a good local error distribution.

II. NUMERICAL MODEL AND ERROR ESTIMATIONS

Given a divergence-free applied current density $\mathbf{J_S}$, the magnetostatic problem reads:

$$\text{div}\mathbf{B} = 0; \quad \mathbf{rot}\mathbf{H} = \mathbf{J_S}; \quad \mathbf{B} = \mu\mathbf{H}, \tag{1}$$

where $\mathbf{B}$ and $\mathbf{H}$ represent respectively the magnetic flux density and magnetic field, the μ stands for the magnetic permeability. Two following potential formulations can be obtained by using the vector potential $\mathbf{A}$ s.t. $\mathbf{B} = \mathbf{rot}\mathbf{A}$ and the scalar potential Ω s.t. $\mathbf{H} = \mathbf{H_S} - \mathbf{grad}\Omega$, where $\mathbf{rot}\mathbf{H_S} = \mathbf{J_S}$:

$$\mathbf{rot}(\frac{1}{\mu}\mathbf{rot}\mathbf{A}) = \mathbf{J_S} \text{ and } \text{div}(\mu\mathbf{grad}\Omega) = \text{div}(\mu\mathbf{H_S}). \tag{2}$$

Let us denote $\mathbf{B}_h$ and $\mathbf{H}_h$ the numerical solutions from each formulations. We define respectively the discretization error in $\mathbf{A}$ and Ω formulation on the domain D by

$$\|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\|_D \text{ and } \|\mu^{1/2}(\mathbf{H} - \mathbf{H}_h)\|_D. \tag{3}$$

Our proposed equilibrated estimators consist in finding an equilibrated flux reconstruction $\sigma_\mathbf{B}$ for $\mathbf{A}$ formulation (or $\sigma_\mathbf{H}$ for Ω formulation) in Raviart–Thomas–Nédélec subspace over the mesh $\mathcal{T}_h$. With this flux reconstruction $\sigma_\mathbf{B}$ (or $\sigma_\mathbf{H}$), we can establishes the estimator verifying the reliability and local efficiency, i.e.

$$\|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\|_D^2 \leq \sum_{T \in \mathcal{T}_h} \|\mu^{-1/2}(\sigma_\mathbf{B} - \mathbf{B}_h)\|_T^2.$$

and

$$\|\mu^{-1/2}(\sigma_\mathbf{B} - \mathbf{B}_h)\|_T \leq \|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\|_{patch(T)} + \eta_{osc}.$$

where η_{osc} is the local data oscillation, and the notation $patch(T)$ is the neighborhood of the element T (same result for $\sigma_\mathbf{H}$ case). Local problems should be resolved to obtain the flux reconstruction $\sigma_\mathbf{B}$ or $\sigma_\mathbf{H}$, see [5] and [6] for details. It should be noted that, different from the residual type estimator, there is no unknown constant C here.
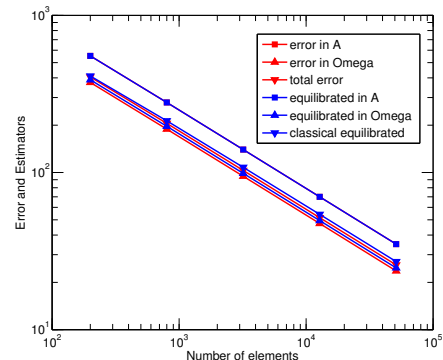

Fig. 1. Errors and equilibrated estimators on uniformly refined meshes.

III. Numerical application

It has been discussed in our previous work [3], the residual type estimator can evaluate the local error distribution for two potential formulations, while the classical equilibrated one cannot separate the individual information of each formulation. However, the classical equilibrated one can give a guaranteed bound for the sum of two formulations. To show the improvement from our proposed estimator, we consider the same the academic problem (Rubinacci cube) with an analytic solution [7]. Five successive levels of refined uniform meshes are considered.

(a) Error on $\mathbf{A}$

(b) Error on Ω

(c) Equilibrated estimator on $\mathbf{A}$

(d) Equilibrated estimator on Ω

(e) Residual estimator on $\mathbf{A}$

(f) Residual estimator on Ω

(g) Total error on $\mathbf{A}$ and Ω
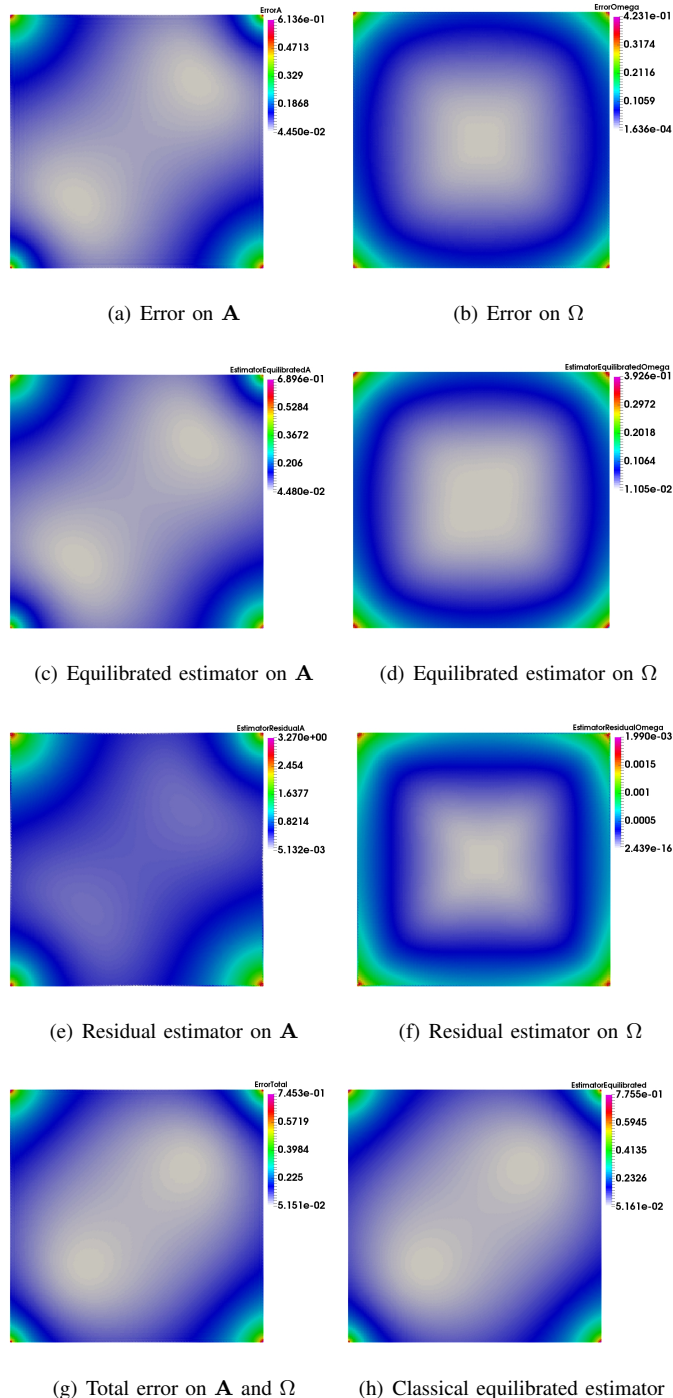
(h) Classical equilibrated estimator

Fig. 2. Spatial distributions of the errors and estimators.

Fig. 1 displays the global errors and the various equilibrated estimators as a function of the number of mesh elements. It can be seen that all the estimators have the same convergence order than the errors. Moreover, similar as the classical equilibrated estimator which can give a sharp error bound (indeed almost coincides with the total error), our proposed equilibrated estimators (both in $\mathbf{A}$ and in Ω) yield almost indistinguishable value for the discretization error of each formulation (error in $\mathbf{A}$, error in Ω).

In Fig. 2, the spatial distribution of different errors and of estimators are shown. Fig. 2(a) and Fig. 2(b) are respectively the exact error in formulation $\mathbf{A}$ and Ω. Fig. 2(c) and Fig. 2(d) are our proposed equilibrated estimator for each correspondent formulation. A good match between (a) and (c), (b) and (d) can be observed. Fig. 2(e) and Fig. 2(f) are the residual type estimators for each formulation, a relative match between (a) and (e), (b) and (f) can be found, but the value of estimators does not have any senses. At last, Fig. 2(g) and Fig. 2(h) are respectively the sum of total error by two formulations and the classical equilibrated estimator. As we explained in our previous work [3], the classical one can give a good accordance for the total error distribution, but cannot separate the information for each formulation.

IV. Conclusion

The equilibrated type error estimators based on the flux reconstruction have been developed for both $\mathbf{A}$ and Ω potential formulations in magnetostatic problems. A comparison with the residual type, as well as the classical equilibrated type has been carried out with numerical examples. For each formulation, similar to the residual type, our proposed estimators can give good accordance with the correspondent error distribution. Furthermore, similar to the classical equilibrated type, we can also give a guaranteed global bound for each formulations. In addition, in order to carry out our equilibrated estimator, we do not need the full resolution of two formulations as the classical equilibrated one, all the estimators can be computed based on the flux reconstruction on local problems. Our estimators are shown to be more efficiency to evaluate the discretization error.

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